A Relationship Between Parametric Resonance and Chaos

R. Kobes[*]

Department of Physics, University of Winnipeg,
Winnipeg, Manitoba, R3B 2E9 Canada

S. Peleš[†]

Department of Physics, University of Manitoba,
Winnipeg, Manitoba, R2T 2N2 Canada

Abstract

In this paper we study two types of exponential instability – parametric resonance and chaos. We show that a given equation may produce chaos or parametric resonance, depending how the problem is defined. In so doing we establish a relationship between the Floquet indices (associated with parametric resonance) and Lyapunov exponents (associated with chaos).

[*]Electronic address: randy@theory.uwinnipeg.ca; Winnipeg Institute for Theoretical Physics, Winnipeg, Manitoba, R3B 2E9 Canada

[†]Electronic address: peles@theory.uwinnipeg.ca; Winnipeg Institute for Theoretical Physics, Winnipeg, Manitoba, R3B 2E9 Canada

I. INTRODUCTION

Parametric resonance is a phenomenon that occurs in various cosmological and high energy physics models. It manifests itself as a rapid growth of a physical field at an exponential rate. Recently this phenomenon has been used to explain some physical processes such as reheating in the early universe [1, 2] and in phase transitions in disordered chiral condensates [3]. At the same time a lot of attention has been given to the study of chaotic systems, i.e. systems whose trajectories in phase space diverge exponentially, but at the same time remain within a bounded region. As both types of systems are described by an exponential type of instability one might expect a relationship between the two, and in this paper we investigate quantitatively just such a relationship. We show that for a system exhibiting parametric resonance it is possible to construct an equivalent chaotic system, although the converse is not guaranteed.

A. Floquet Index

From the general theory of differential equations we know that any second-order linear differential equation

$$\frac{d^2y}{dt^2} + f(t)\frac{dy}{dt} + g(t)y = 0, \tag{1}$$

will have two linearly independent solutions. According to Floquet's theorem [4] if $f(t)$ and $g(t)$ are functions periodic in t with a period T, then those solutions will have form:

$$y(t) = e^{\mu t}P(t), \tag{2}$$

where $P(t)$ is periodic function with period T, as well. Therefore, stability of the solution (2) is entirely determined by the exponent μ, which is also called Floquet exponent or Floquet index. There is no general procedure for estimating Floquet exponent, however there are a lot of particular cases such as the Mathieu equation where an extensive analysis of the Floquet indices has been done.

B. Lyapunov Exponents

Lyapunov exponents are a quantitative indication of chaos, and are used to measure the average rate at which initially close trajectories in phase space diverge from each other. The Lyapunov exponent is usually defined as:

$$\lambda_i = \lim_{t \to \infty} \frac{1}{t} \ln \frac{\epsilon_i(t)}{\epsilon_i(0)} \tag{3}$$

where $\epsilon_i(t)$ denotes the separation of two trajectories, and the index i denotes the direction of growth in phase space or Lyapunov directions [5]. If at least one Lyapunov exponent is positive, then trajectories of the system will separate at an exponential rate and we say that system is chaotic. This will manifest itself in a high sensitivity to a change in the initial conditions. Since the motion of chaotic systems is performed within a bounded region in phase space, besides an exponential stretch some kind of fold of the trajectory must occur. Because of that, the Lyapunov directions i in phase space are not constant in time. They tend
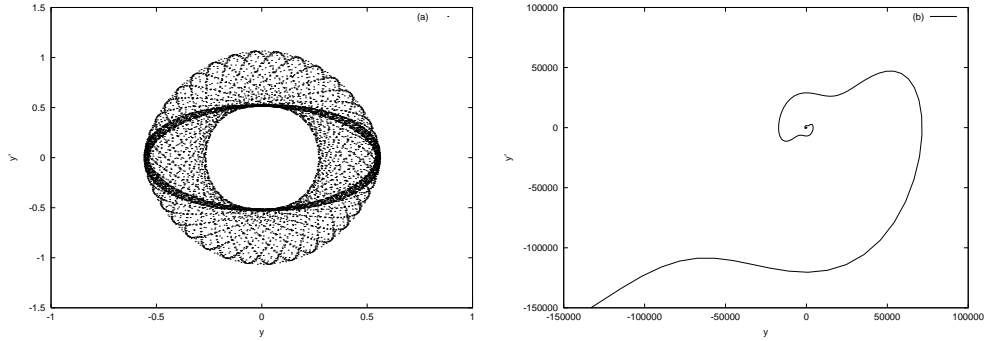
FIG. 1: Phase space trajectory for the Mathieu equation. (a) $A = 2.5$ and $q = 1$, (b) $A = 1$ and $q = 1$.

to rotate, and at the same time to collapse towards the direction of the most rapid growth. A powerful and very general algorithm for determining Lyapunov exponents is presented in Ref. [6, 7], and its implementation, which we use for our calculations, in Ref. [5].

The algorithm is based on an analysis of the evolution of an orthonormal basis in phase space along a trajectory. In the linear approximation directions of growth in phase space will remain orthogonal, therefore allowing one to measure the growth directly. In Ref. [5] it is suggested to describe evolution of the orthonormal basis by linearized equations and use a small enough time step so this approximation remains valid, and so we can neglect the effect of the Lyapunov directions collapsing. After the growth was measured the basis is re-orthonormalized, and the procedure is repeated for the next small enough time interval. If the limit of Eq. (3) exists, then the average of the measured growths should converge to its asymptotic value. This is a robust numerical algorithm and it works for almost every problem whose governing equation is known. It is not suitable though for unbounded systems that diverge exponentially, since the limits of numerical accuracy may be exceeded before the average growth starts converging.

II. MATHIEU EQUATION

Perhaps the simplest model that exhibits parametric resonance is the Mathieu equation (4), which was originally used to describe small oscillations of a pendulum with vertically driven base.

$$y'' + (A - 2q\cos 2t)y = 0 \tag{4}$$

This is a Floquet type equation with two parameters A and q, and it has a solution of the form of Eq. (2). The value of the Floquet index μ depends on the equation's parameters. For certain values of A and q (e.g. $A = 2.5$ and $q = 1$) Floquet exponents would be purely imaginary, meaning that the solutions of Eq. (2) will both be periodic. Therefore, the solution will be stable, and its trajectory in phase space will remain within a bounded region (see Fig. IIa). Otherwise, both Floquet exponents will be purely real, and one of them hence positive. A solution with a positive Floquet exponent is unstable and grows exponentially (see Fig. IIb). In some physical models such growth of the field y can be interpreted as a massive production of certain particles. This is also referred to as parametric resonance.

The rate of exponential growth of the solution (i.e. a positive Floquet exponent) can be determined from the graph for $\log|y|^2$ plotted against time t. Since the term in the
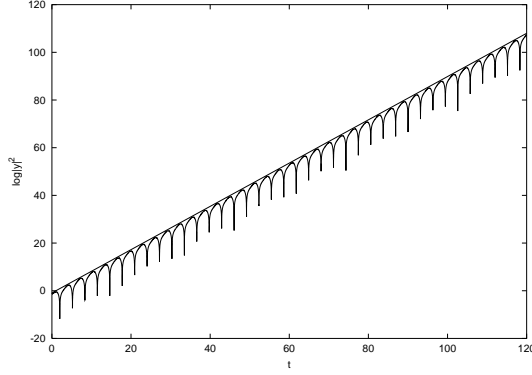
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FIG. 2: Exponential growth of the solution for the Mathieu equation. The Floquet exponent is estimated from the slope to be $\mu = 0.453 \pm 0.003$
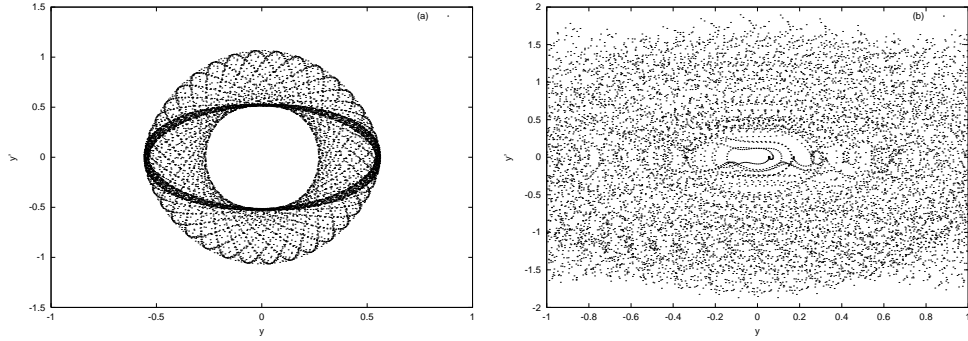

FIG. 3: Phase space trajectory for the Mathieu equation with appropriate winding condition. The periodic solution (a), $A = 2.5$ and $q = 1$, remains unchanged, while the unstable solution (b), $A = 1$ and $q = 1$, becomes chaotic.

solution containing a positive exponent is dominant, the slope of envelope of the graph will yield numerical value of 2μ. From Fig. (II) it is found that $\mu = 0.453 \pm 0.003$, where the parameters are chosen to be $A = 1$ and $q = 1$. Regions of stability in parameter space of the Mathieu equation have been very well studied (see, for example, Ref. [8]). There are bands of stability and instability in the parameter space, and their boundaries are continuous curves.

In analogy with interpreting y as an angle, we impose suitable "winding" conditions on the solution of Eq. (4) so that it always stays within segment $[-1, 1]$. There is no physical motivation to interpret y as an angle though, unless it stays within the limits of a small angle approximation. With this additional restriction imposed, both stable and unstable solutions are bounded, so parametric resonance does not occur. The stable solution remains periodic and exhibits the same behavior as before (see Fig. II). The unstable solution, on the other hand, instead of parametric resonance, exhibits chaotic-like behavior (see Fig. II) which manifests in high sensitivity in change of initial conditions. For this solution we estimated the Lyapunov spectrum, and found the positive Lyapunov exponent to be $\lambda_1 = 0.453 \pm 0.001$, which is the same as the Floquet exponent. This result could be anticipated because the first Lyapunov direction always point to the direction of the fastest growth in phase space. For the Mathieu equation this growth is entirely described by the solution with a positive Floquet exponent. The linearization procedure in the algorithm
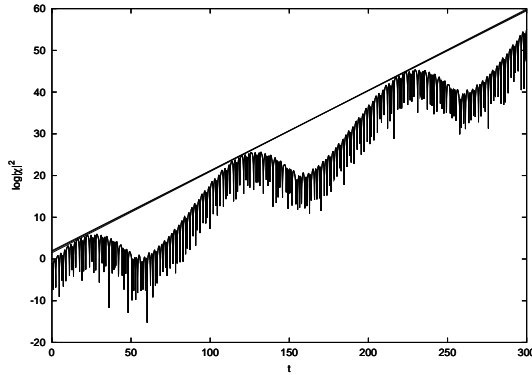
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FIG. 4: Plot of $\log|\chi|^2$ vs. t shows how the Floquet exponent changes during the time. The average Floquet exponent is estimated from the graph to be $\mu_\chi = 0.0973 \pm 0.0007$.

for the Lyapunov exponents calculation will in a sense "unwind" the trajectory, so that the exponential divergence measured by the Lyapunov exponent has to be the same as that described by the Floquet index.

III. PARAMETRIC RESONANCE MODEL

If the parameters of the Mathieu equation Eq. (4) are not constant, but rather some functions of time, then the solution will eventually switch between regions of stability and instability in parameter space, and therefore phases of quasiperiodicity and exponential growth will interchange during that time. Here is a somewhat simplified system of equations which illustrates such behavior. Eqs. (5, 6) may be used to describe the decay of ϕ-particles into χ-particles.

$$\ddot{\chi} + H\dot{\chi} + (m^2 + g\phi^2)\chi = 0 \tag{5}$$

$$\ddot{\phi} + (A - 2q\cos 2t)\phi = 0 \tag{6}$$

Eq. (5) is a Floquet–type of equation, with parameters m and g set near the boundary between the stability and instability regions. Eq. (6) is a Mathieu equation which is coupled to Eq. (5). If we set the parameters A and q so that Eq. (6) has a periodic solution, then the term $g\phi^2$ will periodically drive Eq. (5) between its stability and instability regions, and therefore its Floquet exponent will change periodically in time. This is shown in Fig. III, where $\log|\chi|^2$ is plotted versus time. Although the Floquet exponent changes periodically in time, the system spends more time in a region of instability, and parametric resonance occurs. The average value of the Floquet exponent has a positive real part, and its numerical value can be estimated directly from Fig. III.

Now, let us impose the same "winding" conditions like that for the Mathieu equation, so that $\chi \in [-1, 1]$. As before, parametric resonance will not occur, but the field χ will exhibit chaotic–like behavior instead. In order to find the Lyapunov exponent spectrum for this system we need to perform our calculation in 5-dimensional phase space $(\dot{\chi}, \chi, \dot{\phi}, \phi, t)$. We found two exponents in the spectrum to be positive, and their sum to be $\lambda_1 + \lambda_2 = 0.0973 \pm 0.0006$, which agrees with the value for average Floquet exponent.

Again, this is an expected result, considering the algorithm for estimation of the Lyapunov exponents. The sum of all positive Lyapunov exponents is an average rate of exponential
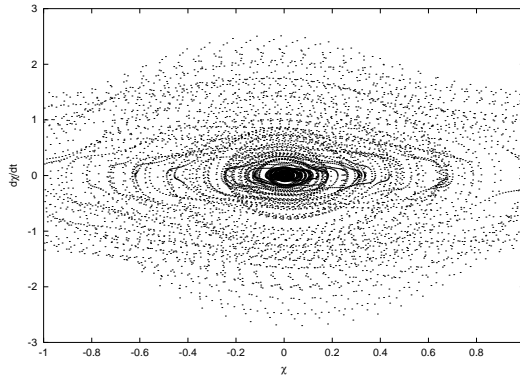
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FIG. 5: Phase space projection of the chaotic pseudo-attractor of Eq. (5).

divergence of the solution of Eq. (5). This should be the same as the rate estimated from Fig. III, as the slope there is determined by the average value of Floquet exponent.

If we, however, substitute a chaotic solution of the Mathieu equation into Eq. (5), the system will exhibit a very complex behavior. The system will chaotically switch between stability and instability regions so it will be impossible to predict any kind of resonant behavior due to a high sensitivity to change in initial conditions. Furthermore, Eq. (5) will not be a Floquet equation any more, and its solution will not have the simple form of Eq. (2).

IV. CONCLUSIONS

We demonstrated here that parametric resonance and chaos are two types of exponential instability which are mutually exclusive but related. Starting from a model that satisfies Floquet's theorem and is in a region of parameter space which is exponentially unstable (with a positive Floquet index), we showed that imposing a "winding" type of boundary condition on the field to restrict it to lie within a certain range leads to a model exhibiting chaos, and hence with at least one positive Lyapunov exponent. A quantitative measure of the exponential divergence rate in the two related models of the cases we studied shows that the Floquet exponent is equal to the Lyapunov exponent. Some extensions and applications of this correspondence between these two types of instabilities are currently being studied.

[1] L. Kofman, A. Linde, and A. Starobinsky, Phys. Rev. Lett. **73**, 3195 (1994).
[2] D. Kaiser, Phys. Rev. **D56**, 706 (1997).
[3] D. Kaiser, Phys. Rev. **D59**, 117901 (1999).
[4] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics*, Part I, p. 557 (McGraw–Hill, New York, 1953).

[5] A. Wolf, J. B. Swift, H. L. Swinney, and J. A. Vastano, Physica **D16**, p. 285-317 (1984).

[6] G. Benettin, L. Galgani, A. Giorgilli, and J. Strelcyn, Meccanica Vol. **15**, No. 1, p. 9-19 (1980).

[7] I. Shimada and T. Nagashima, Prog. The. Physics Vol. **61**, No. 6, p. 1605-16 (1979).

[8] M. Abramovitz and I. A. Stegun, *Handbook of Mathematical Functions*, p. 727, (New York, Dover, 1965).